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An inquiry into the multiproduct firm

Gary Glen Swenson
Iowa State University

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Gary Glen Swenson

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CHAPTER I. INTRODUCTION

> Economists are beginning to realize that they have built a rather elaborate edifice on rather insubstantial, narrow foundations.
>
> Robert J. Heilbroner (1974)

The 1960's seemed to be the decade in which economics came of age as a science. That period saw a long interval of sustained, uninterrupted growth, widely attributed to the "New Economics" in which government took an active role in influencing the course of the economy. This success seemed to result from the fact that economists had injected a certain amount of exactness into their discipline. Forecasting models could be built which would simulate the economy and predict price movements, growth rates, and unemployment levels. Given these predictions, the accepted theory seemed capable of prescribing the correct remedies.

But the above scenario does not have a happy ending. By the start of the 1970's, things had started to fall apart. The forecasting models increasingly errored in calling economic turns, and the policies that accepted theory said would work seemed to have lost some of their effectiveness. By the mid-1970's high unemployment and a substantial rate of inflation, phenomena which were at one time deemed mutually exclusive, were hopelessly intertwined. Increasingly, the accepted theory came to be questioned.

The emphasis in economics in the 1960's (if not in the entire post-World War II period) had been on macroeconomics--economics of the large. The relevant variables here were the level of employment, Gross National Product, and the price level. But these aggregate variables were derived by summing over individual units, so the aggregate theory was no better than the theory explaining these individual economic units. For example, Nobel laureate Kenneth Arrow has stated that "The weakness of inflation theory goes right down to the micro level, to the theory of price determination at the level of the individual firm" (38, p. 59). Thus one has seen a renewed interest in microeconomics.

Microeconomics--economics of the small--stresses "the study of the particular rather than the general . . ." (1, p. 86). Of interest here is the behavior of individual business firms, consuming units, and markets. Increasingly, economists have come to realize that a viable microeconomic theory is a prerequisite to understanding and explaining macroeconomic phenomena.

What is the state of microeconomic theory? Unfortunately, changes in micro-theory have not kept pace with changes in the structure of the economic system. Adam Smith's invisible hand (circa 1776), the Marshallian scissors of supply and demand (circa 1890), and the marginal analysis of the 1930's seemed incapable of explaining twentieth-century situations.[1] One writer, for example, noted that value theory (microeconomics) failed "to give a realistic account of price formation in modern industrialized economies" (17, p. 3). It has been noted that the Depression of the 1930's led to "drastic modifications in the orthodox theory of prices" (34, p. 73). The feeling now exists that perhaps once again micro-theory needs some of those "drastic modifications."

The business firm is at the center of microeconomics. It buys inputs in the factor markets or extracts them from their natural state, transforms these inputs into products and services, and sells these to final consumers or other firms. The standard assumption is that the goal of the firm is to maximize profits. In the growing economy of the 1960's it was at times true that companies "made money in spite of themselves" because of growing aggregate demand and relatively cheap resources. But a time of slow (or nonexistent) economic growth and high input costs requires firms to look carefully at the myriad of decisions they make.

The popular literature of the mid-1970's cited numerous examples of firms re-evaluating their product lines (see for example, 39) and pricing policies (32). A relevant question

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[1]Micro-theorists are not to be completely derided. One observes important contributions in welfare economics and general equilibrium analysis to cite several areas. See (33) and (4) for some examples of this work.

is whether or not some model or criteria exists to explain observed behavior or to which a firm can appeal for help in choosing policies which will enhance the position of the firm.

The purpose of this work is a more comprehensive look at the decision variables that are relevant to a business firm. The typical modern firm is multiprocess and multiproduct. Most literature that exists on the theory of the firm is concerned with choice of technology and choice of output levels, along with changes in these choices in response to factor price changes. For the sake of relevance, a different approach is required. The firm should be viewed as an offer maker; the decision variables available to it being those which it can manipulate to vary its offer. The firm must make these decisions, cognizant of the fact that interdependencies exist in production, in sales, and also between these two functions.

There exists in the economics literature fragments of multiproduct firm theory. This literature is reviewed in Chapter II. Chapter III presents a generalized model of a firm that is representative of those in a modern industrialized society. Chapter IV discusses how the model can be used by a firm to increase profitability and also limitations of the model.

CHAPTER II. THE STATE OF THE ART

The purpose of this chapter is to review various aspects of existing efforts that purport to be theories of the business firm and to point out some of their deficiencies. First of all, let the following be taken as the definition of the business firm (22, pp. 595-6):

> The firm is a production and sales organization controlled by a group or individual such that at least one factor of production is allocated over the whole organization by the control group or individual.

In other words, there is at least one element of commonality to all parts of the firm. One such element that comes to mind is the capital budgeting unit.

The firm we are interested in faces a downward sloping demand curve for some or all of its products. Thus it is a monopoly firm in the broad sense that it is the sole producer of its own unique product.[1] It is reasonably easy to motivate a downward sloping demand curve. One writer noted (26, p. 92):

> If there are any differences in the product or services offered which allow the seller to believe that some of his buyers are attached or loyal to him or prefer him for any reason to other sellers of the 'same' (read: similar) goods, the seller may have a choice of possible prices. (parentheses in original)

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[1]The interested reader is referred to (35, p. 5) for further discussion of monopoly in this context.

That writer goes on to state that in order to get a tilted demand curve, "It suffices that some customers have preferences for certain products or certain sellers, and that these preferences are of different intensities" (26, p. 95).

Another writer noted that the assumption of a downward sloping demand curve could be justified because (35, p. 90):

> . . . not all the customers, who are attached in varying degrees to a particular firm by the advantages which it offers them, will immediately forsake it for a rival who offers similar goods at an infinitesimally smaller price.

The development of the theory of the business firm dates back to Riccardo and Malthus. Their firm was an English wheat farm. Inputs were land, labor, and capital; the output was a single, homogeneous commodity. The production function, or the relationship between inputs and outputs, was viewed broadly and intermediate processes and products were ignored. This arrangement yielded such results as the Law of Diminishing Returns and the associated U-shaped cost curves (12, pp. 1-3).

The above model evolved into a model of entrepreneurial behavior and was used to explain entrepreneur decisions. It is still commonly presented as a theory of a business firm in intermediate microeconomic theory texts.[1] For a modern industrialized economy, however, such a model might not ask a lot of important questions. The typical modern firm is

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[1]See for example (18), or (40).

multiprocess and multiproduct. Even if the simpler analysis applies to each process, it can't be used for the firm as a whole. Apparently these processes are related and it is advantageous to the firm to engage in all of them, rather than to have separate firms at each stage of the production cycle. Also, to assume that a firm sells only one product is a fiction. With many products, the sales of any one product potentially affects the sales of all other products. Thus the multiproduct, multiprocess firm can not be analyzed by looking at each product or process separately, but must be analyzed in toto.

There are bits and pieces of such analysis existing in the economic literature. Unfortunately, all of these efforts fall short of being a complete theory of the firm. Many of them have concentrated on a firm which was a perfect competitor (see for example 20, or 21). However, the interest here is a firm with a downward sloping demand curve, thus ruling out the perfect competitor.

By introducing the possibility of a downward sloping demand curve, one makes several new decision variables available to the firm. The firm facing such a demand curve has some control over the price it charges--it is not a price taker. Since the interest here is the decision variables available to the firm, and also since relevance is a desirable goal, we will consider price as one decision variable available to the firm. Therefore, the firm can affect its volume of sales by changing the prices it charges. But if the firm can alter sales by changing price, it has two more devices at its disposal to influence sales: the product itself and promotion of its products. Because of this, one must look not only at the production side of the firm but also the selling side.

The bits and pieces of multiproduct firm literature fall into a rough dichotomy--part analyze the selling effort put forth by the firm and part analyze the production side of the firm. With respect to the selling side, cognizance of the phenomena of firms altering their product and promoting it is a relatively recent event in economics, dating from the 1930's. Prior to that time, emphasis was centered on two market structures--perfect competition and monopoly (that term being used here in the more traditional sense of the firm being the sole supplier of a product with no close substitutes). Selling effort was not considered important in either of these. The perfect competitor sold a homogeneous product so that the product variation route was not available to him. Also he had no incentive to advertise or otherwise promote the product because the output of all firms was identical and the individual firm could sell all it wished to sell at the market price. The monopolist had no incentive to improve his product or advertise it because he had no competitors.

Chamberlin (8) was perhaps the earliest writer to formalize the treatment of nonprice variables available to the firm. He showed profit maximizing adjustment of advertising expenditures, product improvement, and price. His analysis was for a single product firm, however, and his handling of the production side of the firm followed the simplistic approach of the Riccardo-Malthus model. Chamberlin's approach also implicitly assumes imperfect information. Sales promotion and advertising are only profitable if the buyer is not perfectly informed. The fact that the seller is imperfectly informed is reflected by the introduction of the big D and little d demand curves--the objective and the subjective demand curves, respectively.[1] Machlup (26) and Dorfman and Steiner (14) also addressed themselves to the problem of the optimal level of advertising and product adjustment, but again restricted their research to that of a single product firm.

Selling efforts involving firms with more than one product have been investigated by several authors. Clemens (9) noted that "what the firm has to sell is not a product, or even a line of products, but rather its capacity to produce" (9, p. 2). The firm is viewed as facing different markets, each market associated with a different product. The firm should expand

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[1]See (8, pp. 90-94) for a complete discussion.

its product line into new markets as long as price exceeds marginal cost. He cast his analysis in a form similar to the Robinsonian price discrimination case.[1] One problem with this is that the same marginal cost curve was used for all the products the firm produces. It is doubtful that all of the products of a firm have similar cost structures.

Coase (10) and Bailey (5) also inquired into the pricing policy of a firm selling several products. Coase noted that if a firm sells several products, then either the costs of production are interrelated, the demands are interrelated, or both costs and demands are interrelated. Using a two product firm, he worked through the effects of several shocks to the firm (in the form of a tax and an autonomous shift in demand) when these interrelationships occur. The Bailey analysis parallels the Coase approach to a great extent.

One problem with most of the foregoing works is that even though they give lip service to price, product, and promotion being the decision variables, much of the work is still carried out in terms of quantity adjustments. This reflects the lingering influence of the earlier emphasis on perfect competition. Since the perfect competitor was a price taker, his only adjustment variable was quantity.

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[1]See (35, pp. 179-202).

The most sophisticated treatments of the selling side of the firm are those by Scitovsky (37) for the single product firm and Holdren (23) for the multiproduct firm. Taking explicit account of the fact that the firm tries to manipulate the position of the demand curve it faces, Holdren introduces the term "sales function" (22, p. 100). This function has quantity as the dependent variable and price and nonprice dimensions of the offer as the independent variables. A multiproduct firm selling n products then has n sales functions of the following type:

$$
\begin{aligned}
q_1 &= q_1(p_1, p_2, \ldots, p_n; a_1, a_2, \ldots, a_m) \\
q_2 &= q_2(p_1, p_2, \ldots, p_n; a_1, a_2, \ldots, a_m) \\
&\;\vdots \\
q_n &= q_n(p_1, p_2, \ldots, p_n; a_1, a_2, \ldots, a_m)
\end{aligned}
\tag{2.1}
$$

where each a_j is some distinct nonprice way of varying the seller's offer for the associated product, i.e., "any activity of the seller which is perceptibly distinct to the buyer is potentially a distinct a_j" (22, p. 101). These sales functions take specific account of interdependencies among products on the demand side by including the prices of all products sold in any one sales function.

Associated with the production and selling activities is the cost function of the firm:

$$C = C(q_1, q_2, \ldots, q_n; a_1, a_2, \ldots, a_m) \tag{2.2}$$

Profit, or the excess of revenue over cost, is equal to

$$\Pi = \sum_{i=1}^{n} p_i q_i - C \tag{2.3a}$$

or equivalently,

$$\Pi = p_1 q_1 + p_2 q_2 + \ldots + p_n q_n - C(q_1, q_2, \ldots, q_n; a_1, a_2, \ldots, a_m) \tag{2.3b}$$

Recognizing the fact that price and other offer variation items are the relevant decision variables, the first order conditions for profit maximization require that all first order partial derivatives be set equal to zero:

$$\frac{\partial \Pi}{\partial p_i} = 0 \qquad i = 1, 2, \ldots, n \tag{2.4}$$

$$\frac{\partial \Pi}{\partial a_j} = 0 \qquad j = 1, 2, \ldots, m \tag{2.5}$$

Carrying out these operations, one obtains

$$\frac{\partial \Pi}{\partial p_1} = q_1 + \sum_{i=1}^{n} \left(p_i - \frac{\partial C}{\partial q_i}\right) \frac{\partial q_i}{\partial p_1} = 0$$

$$\frac{\partial \Pi}{\partial p_2} = q_2 + \sum_{i=1}^{n} \left(p_i - \frac{\partial C}{\partial q_i}\right) \frac{\partial q_i}{\partial p_2} = 0 \tag{2.6}$$

$$\vdots \qquad \vdots$$

$$\frac{\partial \Pi}{\partial p_n} = q_n + \sum_{i=1}^{n} \left(p_i - \frac{\partial C}{\partial q_i}\right) \frac{\partial q_i}{\partial p_n} = 0$$

corresponding to 2.4 and also

$$\frac{\partial \Pi}{\partial a_1} = \sum_{i=1}^{n} \left(p_i - \frac{\partial C}{\partial q_i}\right) \frac{\partial q_i}{\partial a_1} - \frac{\partial C}{\partial a_1} = 0$$

$$\frac{\partial \Pi}{\partial a_2} = \sum_{i=1}^{n} \left(p_i - \frac{\partial C}{\partial q_i}\right) \frac{\partial q_i}{\partial a_2} - \frac{\partial C}{\partial a_2} = 0 \tag{2.7}$$

$$\vdots \qquad \vdots$$

$$\frac{\partial \Pi}{\partial a_m} = \sum_{i=1}^{n} \left(p_i - \frac{\partial C}{\partial q_i}\right) \frac{\partial q_i}{\partial a_m} - \frac{\partial C}{\partial a_m} = 0$$

corresponding to Equation 2.5.

To make economic sense out of these equations, it is necessary to go back and look at the single product case worked through by Scitovsky (37, pp. 247-264). Such a firm has only one sales function:[1]

$$q = q(p, a_1, a_2, \ldots, a_n) \tag{2.8}$$

with the associated cost function

$$C = C(q, a_1, a_2, \ldots, a_n) \tag{2.9}$$

Profit, Π, is again equal to the difference between total revenue, $p \cdot q$, and total cost

$$\Pi = p \cdot q - C(q, a_1, a_2, \ldots, a_n) \tag{2.10}$$

First order conditions for profit maximization require that the first order partial derivatives of Π with respect to all of its arguments are set equal to zero.

$$\frac{\partial \Pi}{\partial p} = p \frac{\partial q}{\partial p} + q - \frac{\partial C}{\partial q} \frac{\partial q}{\partial p} = 0 \tag{2.11a}$$

$$\frac{\partial \Pi}{\partial a_i} = p \frac{\partial q}{\partial a_i} - \frac{\partial C}{\partial q} \frac{\partial q}{\partial a_i} - \frac{\partial C}{\partial a_i} = 0 \tag{2.12a}$$

$$i = 1, 2, \ldots, n$$

Rearrangement of Equations 2.11a and 2.12a yields the following expressions

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[1]The nonprice offer variables in the Scitovsky work are numbered from one to n. That notation is followed here. This is not to be confused with the n products of the Holdren firm.

$$p - \frac{\partial C}{\partial q} = -q \Big/ \frac{\partial q}{\partial p} \tag{2.11b}$$

$$p - \frac{\partial C}{\partial q} = \frac{\partial C}{\partial a_i} \Big/ \frac{\partial q}{\partial a_i} \tag{2.12b}$$

$$i = 1,2,\ldots,n$$

The left-hand side of these equations is the difference between price and marginal cost, where marginal cost here is the change in cost associated with a change in output brought about by a price change or an autonomous and exogenous shift in the demand for the product. Scitovsky calls this difference the profit margin. The right-hand side of these equations represents what Scitovsky calls "variation cost." He defines it as "The cost of improving the seller's offer sufficiently to raise his sales by one unit . . ." (37, p. 248). Each aspect of the seller's offer has an associated variation cost. Equation 2.11b represents the price variation cost and the Equations in 2.12b represent nonprice variation cost.

These equations express the profit maximizing conditions for a single product firm. The firm should adjust various aspects of its offer until the amount which a unit of the product adds to profit, i.e., the profit margin, is equal to the cost of selling that additional unit, i.e., the offer variation cost.

Returning to the initial Holdren equations, 2.6 and 2.7, upon putting them into the Scitovsky framework one obtains (by taking the n^{th} good as representative):

$$p_n - \frac{\partial C}{\partial q_n} = -q_n \Big/ \frac{\partial q_n}{\partial p_n} - \frac{\sum_{i=1}^{n-1} \left(p_i - \frac{\partial C}{\partial q_i}\right)\frac{\partial q_i}{\partial p_n}}{\frac{\partial q_n}{\partial p_n}} \tag{2.13}$$

The left-hand side of 2.13 is the profit margin of the n^{th} commodity and the right-hand side can be interpreted as the price offer variation cost in the multiproduct case. There would be n equations like 2.13.

With respect to nonprice offer variation cost, Equation 2.7 becomes (again using the n^{th} product and the m^{th} aspect of the offer):

$$p_n - \frac{\partial C}{\partial q_n} = \frac{\left[\frac{\partial C}{\partial a_m} - \sum_{i=1}^{n-1} \left(p_i - \frac{\partial C}{\partial q_i}\right)\frac{\partial q_i}{\partial a_m}\right]}{\frac{\partial q_n}{\partial a_m}} \tag{2.14}$$

The left-hand side of 2.14 is again to be interpreted as the profit margin of the n^{th} product and the right-hand side as the offer variation cost of the m^{th} nonprice offer variation. There would be m•n such equations: n products and m offer variations for each product.

A quick perusal of Equations 2.13 and 2.14 shows some of the complications that arise when we consider the multiproduct firm. If the i^{th} product in 2.13 is complementary with the n^{th} product, $\frac{\partial q_i}{\partial p_n}$ is negative, as is $\frac{\partial q_n}{\partial p_n}$. $\left(p_i - \frac{\partial C}{\partial q_i}\right)$ is assumed positive so that the whole second term on the right-hand side is negative. This has the effect of reducing the optimal profit margin on the n^{th} product. Alternatively, if products i and n are substitutes, that second term is positive and the optimum profit margin on the n^{th} commodity is made larger.

Similar interdependencies are apparent when one looks at Equation 2.14. For example, if a nonprice offer variation affects the sale of other products in a positive fashion,[1] this leads to more of its use because several commodities are

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[1]Such as promotion campaigns for a class of food products or promotion of a name such as "At General Electric, progress is our most important product."

underwriting its cost.[1] Neither the Scitovsky nor the Holdren models represent a completely developed theory of the multi-product firm. Both writers develop their models from a generalized cost function, without working out joint production relationships and intermediate processes. Also, both writers concentrate on just the selling aspects of the firm.

There have also been theoretical developments looking exclusively at the production and cost side of the multiproduct firm. The fact that the firm produces several products causes some complications above and beyond those encountered in the analysis of the single product firm. Fixed factors of production, those which can not be changed in amount in the short run, must be explicitly handled. With several products, the possibility exists of transferring these factors among the products, a possibility which doesn't exist in the single product firm. Since each product may be competing for the use

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[1]It is assumed $\sum_{i=1}^{n-1}\left(p_i - \frac{\partial C}{\partial q_i}\right)\frac{\partial q_i}{\partial a_m}$ is positive, otherwise a_m would not be used. In the equation

$$p_n - \frac{\partial C}{\partial q_n} = \left[\frac{\partial C}{\partial a_m} - \sum_{i=1}^{n-1}\left(p_i - \frac{\partial C}{\partial q_i}\right)\frac{\partial q_i}{\partial a_m}\right]\Bigg/\frac{\partial q_n}{\partial a_m}, \qquad \frac{\partial C}{\partial a_m}$$

is the cost of using additional amounts of the m^{th} nonprice offer variation. It is also assumed that this marginal cost is positive and nondecreasing. Thus it can be made larger (more of the m^{th} dimension of the offer used) to offset the positive second term so that the right-hand side still equals the profit margin.

these limited fixed factors, the multiproduct firm can not be viewed as a collection of single product firms.

Central to the analysis of the single product firm is the production function--a relation between output and inputs which expresses the maximum product obtainable from those inputs, given a fixed plant and the existing state of technology.[1] Thus by writing a production function, we assume that we have solved the technical maximization problem--we combine inputs in such a fashion that we obtain the most output. This is acceptable for a single product firm, but for a multiproduct firm what to produce and how much to produce are no longer technical questions but economic ones dealing with the allocation of the firm's resources.

Most studies of the cost and production problems of the multiproduct firm have taken a programming approach, and in particular a nonlinear programming route. Pfouts (30) was perhaps the first to use this approach, introducing constraints to the effect that the production of the several products could not exceed the capacity of some fixed factors and also including a cost of shifting fixed factors from one product to another.

Letting $x_1, x_2, \ldots, x_n$ denote the n different products, y_{ij} the amount of the j^{th} variable factor used in producing

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[1]For a complete discussion of the production function, see (7, pp. 14-15).

the i^{th} product and w_j its associated price, and z_{it} the amount of the t^{th} fixed factor used in producing the i^{th} product, Pfouts' formal problem was as follows:

Minimize

$$\sum_i \sum_j w_j y_{ij} + K(z_{11}, \ldots, z_{np}) + F \qquad (2.15)$$

subject to .

$$\bar{x}_i - f_i(y_{i1}, \ldots, y_{im}; z_{i1}, \ldots, z_{ip}) = 0 \qquad (2.16)$$

$$i = 1,2,\ldots,n$$

and

$$\sum_{i=1}^{n} z_{ir} - Z_r \leq 0 \qquad (2.17)$$

$$r = 1,2,\ldots,p$$

In 2.15, $K(z_{11}, \ldots, z_{np})$ reflects the cost of switching fixed factors, of which there are p. $\frac{\partial K}{\partial z_{ij}}$ is assumed positive and represents the cost of switching a small amount of fixed factor j to the production of product i. F in this equation represents fixed costs. The problem is to minimize costs, 2.15, subject to a given level of output, 2.16, and also subject to the constraint 2.17 that the firm doesn't use more of a fixed factor than is available, Z_r. Pfouts then appeals to the Kuhn-Tucker theorem to show the conditions which must hold for a cost minimum.

Naylor (28) expands the Pfouts model, handling the problem as one of profit maximization instead of cost minimization and allowing a variety of market structures, whereas Pfouts restricts his analysis to perfect competition in the factor markets. Hence Naylor's problem is to maximize profit, the excess of revenue over costs, where costs again include a cost of switching fixed factors, subject to technical constraints in the form of a production function and the availability of fixed factors. Again the author appeals to the Kuhn-Tucker theorem to establish necessary and sufficient conditions for a profit maximization.

None of the approaches presented in this chapter represent a complete or fully relevant theory of the firm. What follows is an attempt to present a model of the business firm which will more closely approach those which operate in a modern industrial economy.

CHAPTER III. A THEORETICAL FRAMEWORK FOR ANALYZING A MULTIPRODUCT FIRM

Preliminary Comments

It was earlier stated that one goal of this work is to take explicit account of decision variables that are relevant to a business firm. Before a formal model is introduced, some preliminary comments are in order.

Traditional (Neo-classical) analysis views the firm as a mechanism that connects factor markets and the markets for final products. The emphasis is on the flows of inputs and outputs because these quantities reflect the impact of the firm upon the markets in which it participates and also how the firm affects resource allocation. One writer summed up this approach by stating (24, p. 178):

> The economizing problem facing the firm . . . is that of deciding how much to produce and how much of various inputs to use in producing this output . . . .

Fixed factors of production are given only an embryonic treatment in this traditional approach. Their return is often viewed as a residual;[1] and if the firm is postulated as being a single product firm, no question of allocating fixed factors is raised.

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[1]See (18, p. 375) for a discussion of this point.

For completeness and realism, it is necessary to go to a mathematical programming approach. Such an approach is inward looking, expressly concerned with resource allocation within the firm. The rates of flows are still important in a programming framework (in the form of activity levels) but in addition the quantities of fixed factors are central to the problem because they become data determining what the firm can and cannot do (13, pp. 201-2). The programming approach can also take full cognizance of interdependencies in both production and sales which are difficult, if not impossible, to predict in classical production theory.

Attention here is centered on the firm in a particular short run. This emphasis should not lead to myopia with respect to the relevance of the model. It must be realized that the firm is moving along a time path, and that ideally the firm would try to maximize profits over some span of time. What the firm is like during any given time period depends on previous decisions. What it inherits in terms of capital stock, work force, product mix, reputation, and other characteristics are all results of past behavior. Given an awareness of this, current decisions must be made in the context of what has happened previously, appreciating the fact that current decisions affect not only current profits but future profitability as well.

The short run--long run distinction is usually based upon the fixity of factors.[1] The short run is characterized by the fact that there is at least one fixed input: its availability is unalterable in the time period under discussion. Thus in the short run the firm is limited in its capabilities by the properties of any fixed factors. An alternative basis (Alchian) for the short run--long run distinction lies in the amount of time that is allowed to elapse between when the production decision is made and when the first output is available. This approach has the advantage that costs of adjusting factors are built into the production decision.

The expression "costs of production" will be taken to mean (2, p. 23):

> The change in (the decision maker's) equity caused by the performance of some specified operation, where, . . . the attendant change in income is not included in the computation of the change in equity.

The realization of profits during a period would thus mean an increase in equity. Alternatively, a loss during a period would mean a decline in the present value of the firm's assets.

The distinction between factors (fixed or variable) is usually extended to costs. Fixed costs are those costs that are independent of the level of output. These costs define

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[1]The short run--long run dichotomy is a convenient and fruitful artifice. For an insightful discussion of this, see (15).

the scale of plant and do not vary in this short run. A variable cost figure results from summing over the amounts spent on variable factors of production. It is therefore dependent on the level of output.

Most discussions of cost stop with the above bifurcation, but a third type of cost exists which can be identified. Holdren labeled this class of costs "discretionary fixed costs" and stated that they are those costs "which are fixed with respect to output variation, but are decision variables within the functional time period known as the short run" (23, p. 33). These costs are akin to fixed costs because they do not vary with the level of output, but they can be set at different levels in the short run. An example cited by Holdren (whose analysis centered on a retail store) was the level of maintenance, which is a function of the entrepreneur's whim and not strictly dependent on the level of output.

It has been mentioned that what is frequently presented as being the theory of production and cost for a firm is actually relating to only one process within the firm. To produce a given product typically requires numerous processes, and this number is compounded when the firm becomes multiproduct.

Suppose that a firm produces n products and that it were possible to write production functions for each product. One would then have the following expressions:[1]

$$
\begin{aligned}
q_1 &= q_1(y_1, y_2, \ldots, y_r; q_2, q_3, \ldots, q_n) \\
q_2 &= q_2(y_1, y_2, \ldots, y_r; q_1, q_3, \ldots, q_n) \\
&\vdots \\
q_n &= q_n(y_1, y_2, \ldots, y_r; q_1, q_2, \ldots, q_{n-1})
\end{aligned}
\tag{3.1}
$$

where $y_1, y_2, \ldots, y_r$ are inputs. With a multiproduct firm, the production of any one product may be completely independent of the production of other products or it may be affected in a positive or negative fashion. Hence

$$
\frac{\partial q_i}{\partial q_j} \lesseqgtr 0 \tag{3.2}
$$

$$
\begin{aligned}
i &= 1, 2, \ldots, n \\
j &= 1, 2, \ldots, n \\
i &\neq j
\end{aligned}
$$

Given these production functions, it is possible to obtain a cost function:

$$
C = C(q_1, q_2, \ldots, q_n) \tag{3.3}
$$

---

[1]This discussion of the behavior of cost and production functions follows (23, pp. 234-251) closely.

where the q's are implicitly functions of the y's.

Several expressions derived from this cost function are of interest. The marginal cost of the i^{th} product, MC_i, is given by

$$MC_i = \frac{\partial C}{\partial q_i} \tag{3.4}$$

$$i = 1, 2, \ldots, n$$

It is well to remember that marginal cost in this context is the change in cost associated with a change in output brought about by a price change or an autonomous and exogenous shift in the demand for the product.

Another expression of interest is the change in the marginal cost of the i^{th} commodity as the production rate of the j^{th} commodity is varied:

$$\frac{\partial MC_i}{\partial q_j} = \frac{\partial^2 C}{\partial q_j \partial q_i} \tag{3.5}$$

$$i = 1, 2, \ldots, n$$

$$j = 1, 2, \ldots, n$$

$$i \neq j$$

If $\frac{\partial MC_i}{\partial q_j} = 0$, then we can say that these two commodities are unrelated in production. If $\frac{\partial MC_i}{\partial q_j} > 0$, they are competing with one another in production, and if $\frac{\partial MC_i}{\partial q_j} < 0$ we say that they are complementary.

Continuing in this vein a little further, more realism can be injected. Almost every firm or plant utilizes intermediate processes whose outputs are not sold. They are consumed by the enterprise and are necessary for the final output. Let the following notation be introduced:

q_i = the amount of the i^{th} product sold by the firm
$i = 1, 2, \ldots, n$

Q = the vector of sold outputs

$= (q_1, q_2, \ldots, q_n)$

x_k = an intermediate product
$k = 1, 2, \ldots, s$

X = the vector of intermediate products

$= (x_1, x_2, \ldots, x_s)$

Y = the vector of inputs

$= (y_1, y_2, \ldots, y_r)$

a_j = a nonprice offer variation
$j = 1, 2, \ldots, m$

A = the vector of nonprice offer variations

The production functions for the various sold outputs might then be written as follows:

$$\begin{aligned}
q_1 &= q_1(Y;\ q_2, q_3, \ldots, q_n;\ X;\ A) \\
q_2 &= q_2(Y;\ q_1, q_3, \ldots, q_n;\ X;\ A) \\
&\vdots \\
q_n &= q_n(Y;\ q_1, q_2, \ldots, q_{n-1};\ X;\ A)
\end{aligned} \tag{3.6}$$

These equations state that each q is a function of the Y-vector of inputs, the levels of output of other products, the X-vector of intermediate outputs, and the A-vector of nonprice offer variations.

In addition to these equations, a set such as the following would exist:

$$\begin{aligned} x_1 &= x_1(Y;\ Q;\ x_2,\ x_3,\ \ldots,\ x_s;\ A) \\ x_2 &= x_2(Y;\ Q;\ x_1,\ x_3,\ \ldots,\ x_s;\ A) \\ &\ \vdots \\ x_s &= x_s(Y;\ Q;\ x_1,\ x_2,\ \ldots,\ x_{s-1};\ A) \end{aligned} \tag{3.7}$$

These state that each x is a function of Y, Q, A, and all other intermediate products besides itself.

Finally, the following set of equations would appear:

$$\begin{aligned} a_1 &= a_1(Y;\ Q;\ X;\ a_2,\ a_3,\ \ldots,\ a_m) \\ a_2 &= a_2(Y;\ Q;\ X;\ a_1,\ a_3,\ \ldots,\ a_m) \\ &\ \vdots \\ a_m &= a_m(Y;\ Q;\ X;\ a_1,\ a_2,\ \ldots,\ a_{m-1}) \end{aligned} \tag{3.8}$$

These state that each individual a_j is the a_j-function of the vector of inputs, the vector of final outputs, the vector of intermediate products, and all other nonprice offer variations.

Given the relationships that exist in 3.6, 3.7, and 3.8, the total cost function of the firm becomes

$$C = C(Q, X, A) \tag{3.9}$$

Hence total cost depends on the Q-vector of all sold outputs, the X-vector of intermediate products, and the A-vector of non-price offer variations.

In addition to the earlier marginal cost relationships, several more can be derived from Equation 3.9. For example, one could consider the effect of an intermediate process on the cost of a final product:

$$\frac{\partial MC_i}{\partial x_k} = \frac{\partial^2 C}{\partial x_k \partial q_i} \tag{3.10a}$$

$$i = 1, 2, \ldots, n$$

$$k = 1, 2, \ldots, s$$

Depending on the sign of this expression, an intermediate process may be competitive with (if 3.10 > 0), independent of (= 0), or complementary with (< 0) the i^{th} final output. If an intermediate process is used, it must on balance be complementary, or it would not pay to use it.

An expression similar to 3.10a can be derived to show more about the causality involved in a cost change. This expression would be

$$MC_{x_k q_i} = \frac{\partial C}{\partial x_k} \frac{\partial x_k}{\partial q_i} \tag{3.10b}$$

$$k = 1, 2, \ldots, s$$

$$i = 1, 2, \ldots, n$$

This expression shows the change in cost attributable to a change in an intermediate process, where the change in the intermediate process is caused in turn by a change in some sold output.

Another relationship that exists is the effect which a nonprice offer variation can have on the marginal cost of a final product:

$$\frac{\partial MC_i}{\partial a_j} = \frac{\partial^2 C}{\partial a_j \partial q_i} \tag{3.11}$$

$$i = 1, 2, \ldots, n$$

$$j = 1, 2, \ldots, m$$

Once again the possibility of competition (> 0), independence (= 0), or complementarity (< 0) exists.

As final point before setting up a formal model, there is a different type of marginal cost concept:

$$MC_{a_j} = \frac{\partial C}{\partial a_j} \tag{3.12}$$

$$j = 1, 2, \ldots, m$$

This represents the change in cost caused by a change in the level of the j^{th} form of nonprice offer variation. This differs from the previous notion of marginal cost because earlier a change in quantity, q_i, was causing the cost change. 3.12

is the marginal cost associated with changing the offer in the a_j direction.

Theoretical Model of a Multiproduct Firm

Central to this discussion of the business firm is the notion of a production process. Such a process is a technical relationship between inputs and output. What is usually referred to as the production function for a firm actually describes a process. Any firm is a collection of processes, and typically inputs go through many processes before a final product emerges.

Consider the following definitions:

x = output of a process

The combining of variable inputs with the fixed factors of production yields the x's.

These x's can be separated into several identifiable subsets.

x_q^n = output of a process that is sold

$n = 1, 2, \ldots, N$

The firm sells N products. Each x_q^n represents a different type of finished product.

x_a^m = output of a process that is a nonprice offer variation

$m = 1, 2, \ldots, M$

Some processes exist solely for the purpose of affecting demands for final products. The x_a^m's are the outputs of those processes. These outputs aren't sold but do affect sales.

x_I^w = nonsold outputs other than intermediate products.

$w = 1, 2, \ldots, W$

The outputs of some processes accrue to the entire firm and are not necessarily used up in time period t. The levels of output from these processes are not dependent on the profit maximizing levels of output of other processes. Examples include investment activities and research and development activities.

x_y^{ri} = output of the r^{th} process that is used as an input in the i^{th} process. These outputs are nonsold like the previous set, but they are allocable to time period t because the levels at which they appear are dictated by the levels of the x_q^n's and x_a^m's that occur in that time period. There are R such processes the produce these intermediate outputs. Hence $r = 1, 2, \ldots, R$.

These intermediate outputs can be used to produce more intermediate products (of which there are R); they can be used to produce the sold outputs (the N final products); they can be used to produce the nonprice offer variations variable in the short run (of which there are M); or they can be used to produce those outputs that are nonallocable to time t (of which there are W). Therefore, i can go from 1 to R + N + M + W.

x_y^r = output of an intermediate process.

This notation will be used when it is not of importance where this output is being used.

$\bar{y}_s$ = a fixed factor of production

$s = 1, 2, \ldots, S$

These represent fixed factors in the traditional sense. They delineate the limits of the firm and the limits of any given process.

$\hat{y}_k$ = a variable input

$k = 1, 2, \ldots, K$

The utilization of these variable inputs depends on the output levels of the processes.

Bringing some of the above definitions together, one obtains the X vector of outputs of processes:

$$X = (x^1, x^2, \ldots, x^R, x^{R+1}, \ldots, x^{R+N}, x^{R+N+1}, \ldots, x^{R+N+M+W}),$$

or, element by element

$$x^1 = x_y^1$$

$$x^2 = x_y^2$$

$$\vdots$$

$$x^R = x_y^R$$

$$x^{R+1} = x_q^1$$

$$\vdots$$

$$x^{R+N} = x_q^N$$

$$x^{R+N+1} = x_a^1$$
$$\vdots$$
$$x^{R+N+M} = x_a^M$$
$$x^{R+N+M+1} = x_I^1$$
$$\vdots$$
$$x^{R+N+M+W} = x_I^W$$

Figure 3.1 is a schematic diagram of a firm, tracing the origins of the x's and their destinations. These x's, the outputs of the processes, result from combining variable inputs with the fixed factors. They fall into one of four categories: 1) some are finished products and are sold; 2) some are non-price offer variations which affect the level of sales; 3) some are used as inputs of other processes; 4) some remain in the firm and are available for later utilization.

The Neoclassical assumption of profit maximization as the goal of the firm will be maintained here. In other words, the firm seeks to make the excess of revenues over costs as large as possible. This maximization takes place over sold outputs at time t. On the revenue side, the firm faces N sales functions, one for each sold output:

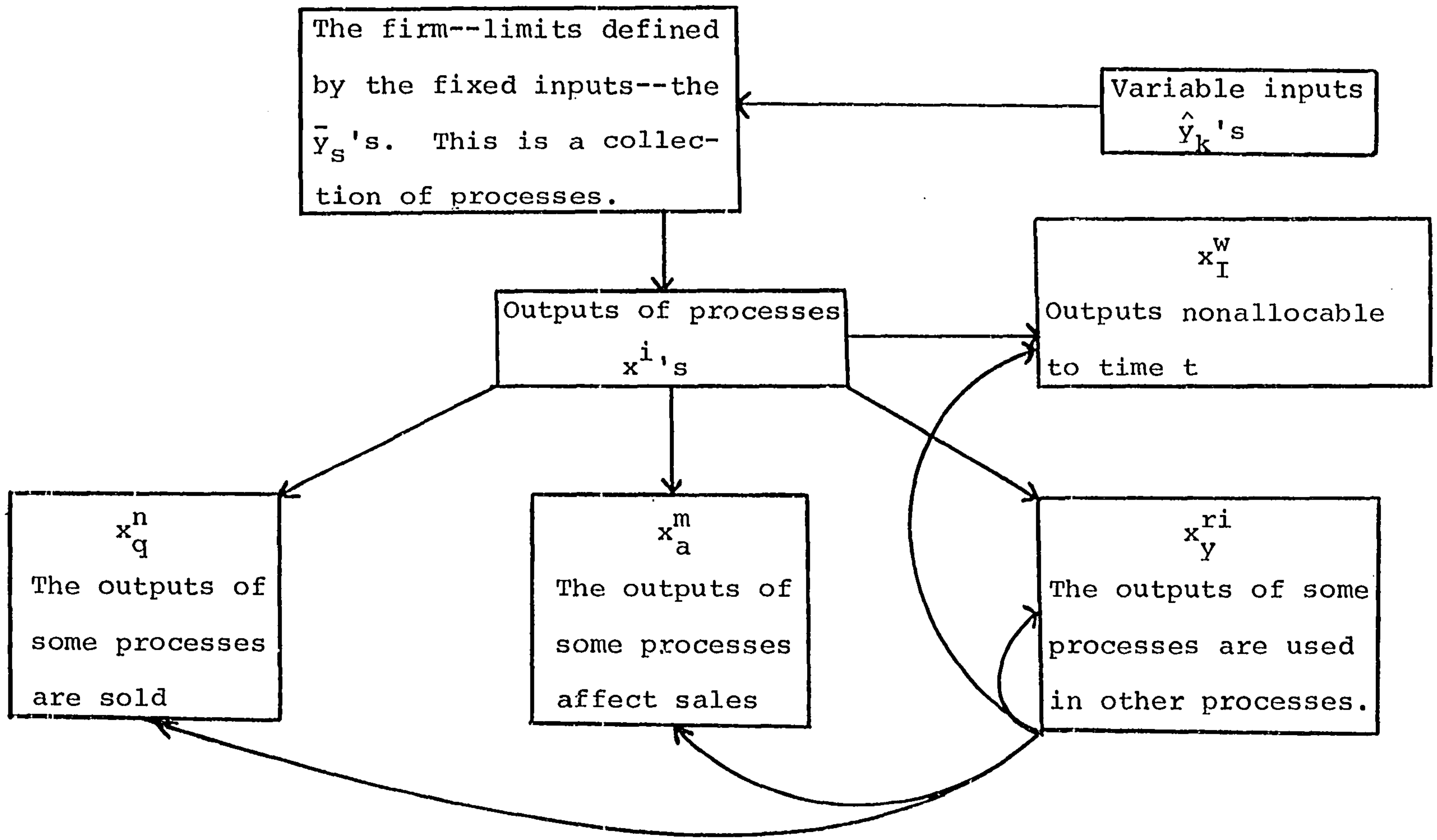

Figure 3.1. Diagram of multiprocess firm

$$x_q^1 = x_q^1(p_1, p_2, \ldots, p_N; x_a^1, x_a^2, \ldots, x_a^M)$$
$$x_q^2 = x_q^2(p_1, p_2, \ldots, p_N; x_a^1, x_a^2, \ldots, x_a^M)$$
$$\vdots \qquad \vdots \tag{3.13}$$
$$x_q^N = x_q^N(p_1, p_2, \ldots, p_N; x_a^1, x_a^2, \ldots, x_a^M)$$

These equations state that the sales of any one commodity may be affected by its own price, the price of any other commodity, and any other aspects of the seller's offer. It is assumed that

$$\frac{\partial x_q^n}{\partial p_n} < 0 \tag{3.14}$$

$$n = 1, 2, \ldots, N$$

i.e., the firm faces a downward sloping demand curve for each of its products.[1] Also,

$$\frac{\partial x_q^n}{\partial p_{n'}} \lesseqgtr 0 \tag{3.15}$$

$$n = 1, 2, \ldots, N$$
$$n' = 1, 2, \ldots, N$$
$$n \neq n'$$

3.15 states that any two products may be complements (if < 0 holds), substitutes (> 0), or independent (= 0) in sales.

---

[1]This assumption is made for the sake of simplicity. It would be easy to introduce a set of constraints on some prices, thus reducing the number of independent variables.

Finally on the sales side

$$\frac{\partial x_q^n}{\partial x_a^m} \lesseqgtr 0 \qquad (3.16)$$

$$n = 1, 2, \ldots, N$$

$$m = 1, 2, \ldots, M$$

states that any change in a nonprice offer variation may affect the sales of any one commodity in a positive manner, in an adverse manner, or not at all.

Total revenue is given by the expression

$$\text{T.R.} = \sum_{n=1}^{N} p_n \, x_q^n \qquad (3.17)$$

which is simply a weighted sum of quantities sold, the weights of course being prices.

Several different types of costs must be considered to arrive at a total cost figure. First of all, the firm incurs fixed costs. These will be split into traditional fixed costs, F_1, and discretionary fixed costs, F_2. As the firm produces outputs, it also incurs variable costs by purchasing variable inputs. Let the following expression be taken to represent the amount spent on variable inputs, i.e., variable costs:[1]

---

[1]This variable cost expression represents costs which result from the maximizing of profit over sold outputs. For completeness one could have a class of costs called discretionary variable costs. This is only mentioned because it is conceivable that some could exist.

$$\text{V.C.} = V(\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_K) \tag{3.18a}$$

The expression $\frac{\partial V}{\partial \hat{y}_k}$ stands for the cost of an additional unit of the k^{th} variable input--the marginal acquisition cost of the factor. If the firm purchases input k in a perfectly competitive market, then

$$\frac{\partial V}{\partial \hat{y}_k} = v_k \tag{3.18b}$$

where v_k is the price of the variable input.

A final type of cost is incurred because of the multiprocess and multiproduct nature of the firm. It becomes necessary to be concerned with how the fixed factors are utilized. The fixed factors can be switched from one use to another, a situation which is not present in single process or single product discussions of the firm, and typically this switching is not costless as retooling and similar adjustments must be made. This gives rise to a final type of cost--switching costs:

$$\text{Switching Costs} = SW(\bar{y}_{11}, \bar{y}_{12}, \ldots, \bar{y}_{1,R+N+M+W}, \ldots, \bar{y}_{S,R+N+M+W}) \tag{3.19}$$

where

$$\frac{\partial SW}{\partial \bar{y}_{si}} > 0 \tag{3.20}$$

$$s = 1, 2, \ldots, S$$
$$i = 1, 2, \ldots, R+N+M+W$$

In this expression $\bar{y}_{si}$ represents the amount of fixed factor s allocated to the process whose output is x^i. The derivative in 3.20 represents the cost of this allocation.

A total cost figure is obtained by combining 3.18, 3.19, and the fixed cost figures:

$$\text{Total Cost} = F_1 + F_2 + V(\hat{y}_k) + SW(\bar{y}_{si}) \tag{3.21}$$

Profit, π, is the excess of revenue over cost and can be written as

$$\pi = \sum_{n=1}^{N} p_n x_q^n - [(F_1 + F_2) + V(\hat{y}_k) + SW(\bar{y}_{si})] \tag{3.22}$$

It is assumed that the firm's goal is to maximize 3.22. The firm doesn't have unlimited freedom in doing so, however. The limits of the firm are defined by the fixed factors. Thus any production is limited by the amount of fixed factors available. The utilization of any fixed factor is a function of variable inputs used with that factor. This utilization can not exceed the availability of the factor, $\bar{y}_s$. This gives rise to a set of constraints such as the following:

$$h_1(\hat{y}_{11}, \hat{y}_{12}, \ldots, \hat{y}_{1K}) \leq \bar{y}_1$$
$$h_2(\hat{y}_{21}, \hat{y}_{22}, \ldots, \hat{y}_{2K}) \leq \bar{y}_2$$
$$\vdots \qquad (3.23)$$
$$h_S(\hat{y}_{S1}, \hat{y}_{S2}, \ldots, \hat{y}_{SK}) \leq \bar{y}_S$$

where $\hat{y}_{sk}$ represents the amount of the k^{th} variable input processed through the s^{th} fixed factor. The h_s function monitors how much $\bar{y}_s$ has been claimed by the various processes as variable inputs are used by the firm.

3.23 is written in a way that departs from the usual treatment of fixed factors. Taking the Pfouts' article (30) as representative, let z_{it} represent the quantity of the t^{th} fixed factor used in the production of the i^{th} product. Z_t represents the total quantity of fixed factor t available. A constraint on the firm with respect to its fixed factors would then be written

$$\sum_i z_{it} \leq Z_t$$

where i indexes the processes which use the t^{th} fixed factor.

By writing the constraints on the availability of the fixed resources in the form embodied in 3.24, previous writers have made several assumptions of which they might not have been aware. First of all, summing assumes that the fixed factors are completely assignable to the outputs. Related to this is the fact that additivity assumes independence and exclusion in the use of fixed factors--what one process uses is not available to be used in other processes.

However, some of the fixed factors might be like public goods, and both of the above conditions would be violated in such cases. Public goods are goods which have the characteristic that ". . . each individual's consumption of such a good leads to no subtraction from any other individual's consumption of that good . . ." (36, p. 387). A common example is a lighthouse, where one ship's use of the beacon in no way diminishes the amount of light available for other ships. This same situation probably holds true for some fixed factors. They are equally available to all processes and the use of such a fixed factor by one process neither diminishes the amount available to other processes nor precludes its use by other processes. This is not to be construed to mean that these public good-type fixed factors don't impose constraints on the firm. They are only available in finite amounts. It is the assigning of these finite amounts that causes problems. Writing 3.23 in its present form allows for these situations.

3.23 will represent the set of constraints imposed on the firm when it attempts to maximize profit. There is also a set of relationships which exist that relate the inputs of any given process to its output. These are analogous to production functions in the sense that their shapes are determined by the fixed factors and the state of technology, and also because we will assume that we obtain the maximum yield from the inputs. These are of the following form:

$$
\begin{aligned}
&g_1(\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_k; x_y^{21}, x_y^{31}, \ldots, x_y^{R1}) = x_y^1 \\
&g_2(\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_k; x_y^{12}, x_y^{32}, \ldots, x_y^{R2}) = x_y^2 \\
&\quad\vdots \\
&g_R(\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_k; x_y^{1R}, x_y^{2R}, \ldots, x_y^{R-1,R}) = x_y^R \\
&g_{R+1}(\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_k; x_y^{1,R+1}, \ldots, x_y^{R,R+1}) = x_q^1 \\
&\quad\vdots \\
&g_{R+N}(\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_k; x_y^{1,R+N}, \ldots, x_y^{R,R+N}) = x_q^N \\
&g_{R+N+1}(\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_k; x_y^{1,R+N+1}, \ldots, x_y^{R,R+N+1}) = x_a^1 \\
&\quad\vdots \\
&g_{R+N+M}(\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_k; x_y^{1,R+N+M}, \ldots, x_y^{R,R+N+M}) = x_a^M \\
&g_{R+N+M+1}(\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_k; x_y^{1,R+N+M+1}, \ldots, x_y^{R,R+N+M+1}) = x_I^1 \\
&\quad\vdots \\
&g_{R+N+M+W}(\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_k; x_y^{1,R+N+M+W}, \ldots, x_y^{R,R+N+M+W}) = x_I^W
\end{aligned}
\tag{3.25}
$$

These state that the output of any process depends on the variable inputs used and any intermediate outputs that are used by the process. The functional forms of 3.25 will be dictated by the state of technology and the stock of fixed factors. These expressions will not initially be built explicitly into the model of profit maximization. They are what might be called side relations in the sense that they state the paths via which the fixed factors are used.

The Mathematics of Profit Maximization

It was previously stated that a nonlinear programming approach would be most fruitful in determining profit maximizing conditions for a multiproduct firm. To an economist, the important question is what will be the characteristics of an optimal output scheme when it is found. These characteristics are summarized in a group of conditions known collectively as the Kuhn-Tucker Theorem.

The Kuhn-Tucker Theorem is a mathematical tool for describing optimality conditions of functions constrained by equalities and inequalities, rather than just equalities as classical constrained optimization requires. The following represents a full generalization of the nonlinear programming approach to the maximizing process in the static sense. Consider the following problem: Find extreme values of a function

$$\psi(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \tag{3.26}$$

where the variables are constrained by inequalities of the following form:

$$\theta_j(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \geq 0 \tag{3.27}$$

$$j = 1, 2, \ldots, J$$

$\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ represent variables under control of the maximizing unit. For a maximization problem, it is necessary to assume that the objective function 3.26 and the constraints 3.27 are concave and differentiable.[1]

The first step in obtaining optimality conditions entails formulating the Lagrangian function:

$$L(\varepsilon_i, \lambda_j) = \psi(\varepsilon_i) + \sum_{j=1}^{J} \lambda_j \theta_j(\varepsilon_i) \tag{3.28}$$

for

$$\varepsilon_i \geq 0 \qquad i = 1, 2, \ldots, n$$

$$\lambda_j \geq 0 \qquad j = 1, 2, \ldots, J$$

To insure the existence of a constrained maximum at $\overset{\circ}{\varepsilon}_i$ and $\overset{\circ}{\lambda}_j$, it is necessary and sufficient that a saddle point exists at the extreme value. To insure the existence of a

---

[1]Optimality conditions have been worked out under less restrictive assumptions concerning concavity. See, for example, (3).

saddle point, it is necessary and sufficient that the following conditions hold:[1]

$$\left.\frac{\partial L}{\partial \varepsilon_i}\right|_{\varepsilon_i=\overset{\circ}{\varepsilon}_i} \leq 0 \tag{3.29}$$

$$i = 1, 2, \ldots, n$$

$$\left.\frac{\partial L}{\partial \varepsilon_i}\right|_{\varepsilon_i=\overset{\circ}{\varepsilon}_i} \cdot \overset{\circ}{\varepsilon}_i = 0 \tag{3.30}$$

$$i = 1, 2, \ldots, n$$

$$\overset{\circ}{\varepsilon}_i \geq 0 \tag{3.31}$$

$$i = 1, 2, \ldots, n$$

$$\left.\frac{\partial L}{\partial \lambda_j}\right|_{\lambda_j=\overset{\circ}{\lambda}_j} \geq 0 \tag{3.32}$$

$$j = 1, 2, \ldots, J$$

$$\left.\frac{\partial L}{\partial \lambda_j}\right|_{\lambda_j=\overset{\circ}{\lambda}_j} \cdot \overset{\circ}{\lambda}_j = 0 \tag{3.33}$$

$$j = 1, 2, \ldots, J$$

$$\overset{\circ}{\lambda}_j \geq 0 \tag{3.34}$$

$$j = 1, 2, \ldots, J$$

[1]These conditions are expressed in several forms in the literature. The statement of them given here comes from (29, p. 151).

Turning to the problem at hand, the firm's problem is to maximize profit, where in the first instance this maximization is constrained by the availability of fixed factors.

Formally, the firm's problem is to maximize

$$\pi = \sum_{n=1}^{N} p_n x_q^n - [(F_1 + F_2) + V(\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_K) + SW(\bar{y}_{si})] \tag{3.35}$$

subject to

$$\begin{aligned} &h_1(\hat{y}_{11}, \hat{y}_{12}, \ldots, \hat{y}_{1K}) \leq \bar{y}_1 \\ &h_2(\hat{y}_{21}, \hat{y}_{22}, \ldots, \hat{y}_{2K}) \leq \bar{y}_2 \\ &\vdots \\ &h_S(\hat{y}_{S1}, \hat{y}_{S2}, \ldots, \hat{y}_{SK}) \leq \bar{y}_S \end{aligned} \tag{3.36}$$

The first step towards obtaining optimality conditions is to put expressions 3.35 and 3.36 into a Lagrangian framework like 3.38. Before this can be done, however, it is necessary to alter several of them in order to take account of certain economic phenomena.

The cost of switching function, 3.19, gives rise to several problems. Recall that $\frac{\partial SW}{\partial \bar{y}_{si}}$ is assumed positive and that it represents the cost of allocating some of fixed factor s to the process whose output is x^i. This cost is an <u>ex ante</u>

cost incurred by the firm in the sense that prior to actual production, a decision must be made as to how the fixed factors will be used. In other words, before production even starts this switching cost is incurred as the fixed factors are allocated to each process in an amount that seems adequate for the anticipated output of that particular process. This means that switching costs are not directly dependent on the rate of output from a given process:

$$\frac{\partial SW}{\partial x^i} = 0 \qquad (3.37)$$

$$i = 1, 2, \ldots, R+N+M+W$$

The switching costs are characterized by what is frequently called "lumpiness" -- they do not change smoothly but instead occur at intervals and in lump sums. If the initial estimate of the amount of a fixed factor required for a process is correct, no more switching costs are incurred by switching that particular factor to the process in question. If the initial allocation proves insufficient, once again a switching cost must be incurred as more of the fixed factor is diverted to that process (assuming, of course, that some is still available).

In order to handle these problems that arise with switching costs, the $SW(\bar{y}_{si})$ function will be subsumed into the discretionary fixed cost group, F_2. Strictly speaking, they aren't discretionary fixed costs because their level will depend to some degree on output variation. However, putting them into F_2 does not damage to the model and essentially simplifies the bookkeeping.

The way in which $V(\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_K)$ changes in response to output changes requires some amplification. Changes in this function occur because the firm employs different amounts of the variable inputs:

$$dV = \frac{\partial V}{\partial \hat{y}_1} d\hat{y}_1 + \frac{\partial V}{\partial \hat{y}_2} d\hat{y}_2 + \ldots + \frac{\partial V}{\partial \hat{y}_K} d\hat{y}_K \qquad (3.38)$$

However, these changes in variable input use are caused by changes in the quantities of sold outputs and changes in the levels of nonprice offer variations. The total change in cost associated with changing the level of production of the n^{th} sold output is equal to:

$$\frac{\partial V}{\partial x_q^n} = \frac{\partial V}{\partial \hat{y}_1} \frac{\partial \hat{y}_1}{\partial x_q^n} + \frac{\partial V}{\partial \hat{y}_2} \frac{\partial \hat{y}_2}{\partial x_q^n} + \ldots + \frac{\partial V}{\partial \hat{y}_K} \frac{\partial \hat{y}_K}{\partial x_q^n} \qquad (3.39)$$

Because of the interdependencies that exist with respect to sales, all prices affect the sales of any given sold output. If the price of the i^{th} sold output changes, a term $\frac{\partial x_q^n}{\partial p_i}$ exists which need not equal zero. This adds another dimension to how costs change. Taking the n^{th} and i^{th} goods as representative, we obtain the following result:

$$\frac{\partial V}{\partial x_q^n}\frac{\partial x_q^n}{\partial p_i} = \frac{\partial V}{\partial \hat{y}_1}\frac{\partial \hat{y}_1}{\partial x_q^n}\frac{\partial x_q^n}{\partial p_i} + \ldots + \frac{\partial V}{\partial \hat{y}_K}\frac{\partial \hat{y}_K}{\partial x_q^n}\frac{\partial x_q^n}{\partial p_i} \qquad (3.40)$$

3.40 holds true for all n goods sold. Since we are interested in the change in variable costs caused by a change in price, $\frac{\partial V}{\partial p_i}$, it is necessary to sum expressions similar to 3.40 over all N goods. This gives rise to

$$\frac{\partial V}{\partial p_i} = \sum_{n=1}^{N}\sum_{k=1}^{K}\frac{\partial V}{\partial \hat{y}_k}\frac{\partial \hat{y}_k}{\partial x_q^n}\frac{\partial x_q^n}{\partial p_i} \qquad (3.41)$$

Finally, from an interpretative standpoint the $V(\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_K)$ function is to be considered net of costs incurred producing the x_I^w's. We are maximizing profit over sold output. Thus we want only those costs that result from those sold outputs, nonprice offer variations, or true intermediate products.

Care must also be taken in interpreting the constraints relating to the availability of the fixed factors, 3.36. The lumpiness discussed earlier relates to the allocation of fixed factors, not their utilization once allocated. The costs associated with switching fixed factors don't directly depend on the outputs of the processes. However, utilization of fixed factors does depend on outputs, and hence on the flows of variable inputs. Because of this,

$$\frac{\partial h_s}{\partial \hat{y}_{sk}} > 0 \tag{3.42}$$

$$s = 1, 2, \ldots, S$$

$$k = 1, 2, \ldots, K$$

if variable factor k passes through the s^{th} fixed factor. Also, the amount of a fixed factor available, $\bar{y}_s$, is to be considered net of the amounts used in producing the x_I^w's. This is so because of the maximization of profit is over sold outputs at time t. We are interested in the first instance in determining optimal levels of the x_q^n's, x_a^m's, and x_y^r's. How the firm determines the x_I^w's is a different type of problem which will be discussed later.

Given the previous remarks, it is now possible to proceed to the Kuhn-Tucker conditions for profit maximization. The relevant decision variables for the firm are those which affect its offer. It will be this set of variables which the firm will adjust in order to maximize profit. In the case at hand, these are the prices of all sold outputs and the levels of the variable nonprice offer variations.

Rewriting the problem after incorporating the change in the cost function, one obtains:

$$\pi = \sum_{n=1}^{N} p_n x_q^n - [(F_1 + F_2) + V(\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_K)] \tag{3.43}$$

subject to

$$\begin{aligned}
&h_1(\hat{y}_{11}, \hat{y}_{12}, \ldots, \hat{y}_{1K}) \leq \bar{y}_1 \\
&h_2(\hat{y}_{21}, \hat{y}_{22}, \ldots, \hat{y}_{2K}) \leq \bar{y}_2 \\
&\vdots \\
&h_S(\hat{y}_{S1}, \hat{y}_{S2}, \ldots, \hat{y}_{SK}) \leq \bar{y}_S
\end{aligned} \tag{3.44}$$

$$p_n \geq 0 \tag{3.45}$$

$$n = 1, 2, \ldots, N$$

$$x^i \geq 0 \tag{3.46}$$

$$i = 1, 2, \ldots, R+N+M+W$$

Upon setting up the Lagrangian function corresponding to 3.28, the following equation results:

$$L(p_n, x_a^m, \lambda_s) = \sum_{n=1}^{N} p_n x_q^n - [(F_1 + F_2) + V(\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_K)]$$

$$+ \sum_{s=1}^{S} \lambda_s [\bar{y}_s - h_s(\hat{y}_{s1}, \hat{y}_{s2}, \ldots, \hat{y}_{sk})] \tag{3.47}$$

The derivation of conditions corresponding to 3.29 to 3.34 results in the following conditions which must be satisfied at $\overset{\circ}{p}_i$, $\overset{\circ}{x^m}_a$, and $\overset{\circ}{\lambda}_s$ in order to insure a constrained maximum:

$$\left.\frac{\partial L}{\partial p_i}\right|_{p_i = \overset{\circ}{p}_i} = p_i \frac{\partial x_q^i}{\partial p_i} + x_q^i + \sum_{\substack{n=1 \\ n \neq i}}^{N} p_n \frac{\partial x_q^n}{\partial p_i} - \sum_{n=1}^{N} \sum_{k=1}^{K} \frac{\partial V}{\partial \hat{y}_k} \frac{\partial \hat{y}_k}{\partial x_q^n} \frac{\partial x_q^n}{\partial p_i}$$

$$- \sum_{s=1}^{S} \lambda_s \sum_{k=1}^{K} \frac{\partial h_s}{\partial \hat{y}_{sk}} \frac{\partial \hat{y}_{sk}}{\partial x_q^i} \frac{\partial x_q^i}{\partial p_i} \leq 0 \qquad (3.48)$$

$$i = 1, 2, \ldots, N$$

$$\left.\frac{\partial L}{\partial p_i}\right|_{p_i = \overset{\circ}{p}_i} \cdot \overset{\circ}{p}_i = 0 \qquad (3.49)$$

$$i = 1, 2, \ldots, N$$

$$\overset{\circ}{p}_i \geq 0 \qquad (3.50)$$

$$i = 1, 2, \ldots, N$$

$$\left.\frac{\partial L}{\partial x_a^m}\right|_{x_a^m = \overset{\circ}{x}_a^m} = \sum_{n=1}^{N} p_n \frac{\partial x_q^n}{\partial x_a^m} - \sum_{n=1}^{N} \sum_{k=1}^{K} \frac{\partial V}{\partial \hat{y}_k} \frac{\partial \hat{y}_k}{\partial x_q^n} \frac{\partial x_q^n}{\partial x_a^m}$$

$$- \sum_{k=1}^{K} \frac{\partial V}{\partial \hat{y}_k} \frac{\partial \hat{y}_k}{\partial x_a^m} - \sum_{s=1}^{S} \lambda_s \sum_{k=1}^{K} \frac{\partial h_s}{\partial \hat{y}_{sk}} \frac{\partial \hat{y}_{sk}}{\partial x_a^m} \leq 0 \qquad (3.51)$$

$$m = 1, 2, \ldots, M$$

$$\left.\frac{\partial L}{\partial x_a^m}\right|_{x_a^m = x_a^{m°}} \cdot x_a^{m°} = 0 \tag{3.52}$$

$$m = 1, 2, \ldots, M$$

$$x_a^{m°} \geq 0 \tag{3.53}$$

$$m = 1, 2, \ldots, M$$

$$\left.\frac{\partial L}{\partial \lambda_s}\right|_{\lambda_s = \lambda_s^°} = \bar{y}_s - h_s(\hat{y}_{s1}, \hat{y}_{s2}, \ldots, \hat{y}_{sK}) \geq 0 \tag{3.54}$$

$$s = 1, 2, \ldots, S$$

$$\left.\frac{\partial L}{\partial \lambda_s}\right|_{\lambda_s = \lambda_s^°} \cdot \lambda_s^° = 0 \tag{3.55}$$

$$s = 1, 2, \ldots, S$$

$$\lambda_s^° \geq 0 \tag{3.56}$$

$$s = 1, 2, \ldots, S$$

Expressions 3.48 to 3.56 represent the conditions that must be satisfied in order for a multiproduct firm to maximize profit. However, these conditions haven't addressed themselves to another problem that arises, and by solving that problem perhaps the results from 3.48 to 3.56 can be improved upon. Before this problem is dealt with, an economic interpretation of the profit maximization conditions will be discussed.

Consider first the nonnegativity restriction on prices:

$$\overset{\circ}{p}_i \geq 0 \qquad (3.57)$$

$$i = 1, 2, \ldots, N$$

This stipulates that the prices which the firm charges must be positive or zero. A question of interpretation arises with respect to the meaning of a zero price. If some output has a zero price, that particular item will be considered a nonprice offer variation. If price is zero, the only remaining decision with respect to the output is the level of utilization. Because of this, a zero price will be taken as a signal to handle the item as an x_a^m.

This results in a restriction that prices charged by the firm have to be positive. It is well to recall that the p_n's are associated with sold outputs. Prior to the profit maximization problem at hand, the firm must make a decision as to what it is going to sell, i.e., exactly what is contained in the set of x_q^n's. It seems reasonable to assume that a firm will only choose items to include in its product line which can be sold at positive prices (This borders on a tautological argument. Selling probably implies a positive price. Otherwise, the firm is giving something away.).

Expression 3.48 represents the condition that must hold if a multiproduct firm has set the price of the i^{th} sold output at the profit maximizing level. It should be noted that the inequality can be done away with and that the strict equality must hold. This is true because expression 3.49 requires that either $\left.\frac{\partial L}{\partial p_i}\right|_{p_i=p_i^\circ}$ or p_i° is equal to zero. Since p_i° is strictly positive, it must follow that $\left.\frac{\partial L}{\partial p_i}\right|_{p_i=p_i^\circ}$ is zero.

A rearrangement of 3.48 makes it more amenable to economic interpretation. Taking the N^{th} good as representative, putting 3.48 into the profit margin framework of the Scitovsky-Holdren model results in:[1]

$$\left(p_N - \sum_{k=1}^{K} \frac{\partial V}{\partial \hat{y}_k} \frac{\partial \hat{y}_k}{\partial x_q^N}\right) \frac{\partial x_q^N}{\partial p_N} = - x_q^N - \sum_{n=1}^{N-1} \left(p_n - \sum_{k=1}^{K} \frac{\partial V}{\partial \hat{y}_k} \frac{\partial \hat{y}_k}{\partial x_q^n}\right) \frac{\partial x_q^n}{\partial p_N}$$

$$+ \sum_{s=1}^{S} \lambda_s \sum_{k=1}^{K} \frac{\partial h_s}{\partial \hat{y}_{sk}} \frac{\partial \hat{y}_{sk}}{\partial x_q^N} \frac{\partial x_q^N}{\partial p_N} \qquad (3.58)$$

$\frac{\partial x_q^N}{\partial p_N}$ represents the change in the amount of the N^{th} product sold as a result of changing its price. Division of 3.58 by this term leaves the expression by itself on the left side and puts the right side on a per unit basis. This operation results in

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[1]See pp. 11-18.

$$p_N - \sum_{k=1}^{K} \frac{\partial V}{\partial \hat{y}_k} \frac{\partial \hat{y}_k}{\partial x_q^N} = - x_q^N \bigg/ \frac{\partial x_q^N}{\partial p_N} - \sum_{n=1}^{N-1} \left(p_n - \sum_{k=1}^{K} \frac{\partial V}{\partial \hat{y}_k} \frac{\partial \hat{y}_k}{\partial x_q^n} \right) \frac{\partial x_q^n}{\partial p_N} \bigg/ \frac{\partial x_q^N}{\partial p_N}$$

$$+ \sum_{s=1}^{S} \lambda_s \sum_{k=1}^{K} \frac{\partial h_s}{\partial \hat{y}_{sk}} \frac{\partial \hat{y}_{sk}}{\partial x_q^N} \qquad (3.59)$$

3.59 can be interpreted term by term from an economics standpoint. On the left side of the equality, $\sum_{k=1}^{K} \frac{\partial V}{\partial \hat{y}_k} \frac{\partial \hat{y}_k}{\partial x_q^N}$ represents the total change in variable costs directly attributable to selling an additional unit of the N^{th} product. p_N represents the price being charged per unit sold of the N^{th} product. The difference between these two is the profit margin on the N^{th} good, i.e., the amount each unit sold contributes to firm profitability.

The right side of 3.59 represents several costs that are incurred by the firm as a result of changing p_N. The first two terms combined represent the multiproduct price offer variation cost. It represents what might be called external costs incurred by the firm as a result of changing p_N. These are external because they reflect how a change in p_N affects the levels of all other sold outputs and their profit margins.

The next term is an additional cost that arises because of a change in p_N. It represents what might be called an internal cost because it relates directly to the utilization of resources within the firm. $\sum_{k=1}^{K} \frac{\partial h_s}{\partial \hat{y}_{sk}} \frac{\partial \hat{y}_{sk}}{\partial x_q^N}$ tells how the utilization of fixed factor s changes because of different employment rates of variable inputs, where the variable input changes are necessitated because x_q^N changed. It is some number representing physical units. λ_s is the Lagrangian multiplier and can be interpreted as the imputed dollar value of a unit of fixed factor s.[1]

$\lambda_s \sum_{k=1}^{K} \frac{\partial h_s}{\partial \hat{y}_{sk}} \frac{\partial \hat{y}_{sk}}{\partial x_q^N}$ is then the dollar value of any change in the utilization of fixed factor s caused by changing x_q^N. Summing over all s fixed factors gives the change in the value of these factors that results from changing x_q^N.

In words, 3.59 states

$$\text{Profit Margin} = \left\{\begin{array}{l}\text{Multiproduct}\\ \text{price offer}\\ \text{variation cost}\end{array}\right\} + \left\{\begin{array}{l}\text{Imputed value of the}\\ \text{change in utilization}\\ \text{of fixed factors}\end{array}\right\}$$

For the profit maximizing adjustment of p_N, the profit margin on the N^{th} good should contribute just enough to offset the

[1]Kantorovich calls these multipliers "objectively determined valuations" which is a very meaningful title. They are objectively determined in the sense that the values which they take on depend on the initial endowments and constraints and result from the mathematics. They are valuations in the sense that they "permit a numerical valuation of the scarcity of the conditions of production, the scarcity of resources, restrictions of equipment, and the strain of the programme" (25, p. viii).

total increase in cost of selling an additional unit.

Additional insight can be gained into the profit maximizing adjustment of price by a different arrangement of 3.59. The terms could be collected in the following manner:

$$p_N \frac{\partial x_q^N}{\partial p_N} + x_q^N = \sum_{k=1}^{K} \frac{\partial V}{\partial \hat{y}_k} \frac{\partial \hat{y}_k}{\partial x_q^N} \frac{\partial x_q^N}{\partial p_N} - \sum_{n=1}^{N-1} \left[p_n - \sum_{k=1}^{K} \frac{\partial V}{\partial \hat{y}_k} \frac{\partial \hat{y}_k}{\partial x_q^n} \right] \frac{\partial x_q^n}{\partial p_N}$$

$$+ \sum_{s=1}^{S} \lambda_s \sum_{k=1}^{K} \frac{\partial h_s}{\partial \hat{y}_{sk}} \frac{\partial \hat{y}_{sk}}{\partial x_q^N} \frac{\partial x_q^N}{\partial p_N} \tag{3.60}$$

Consider the term on the left side of 3.60

$$p_N \frac{\partial x_q^N}{\partial p_N} + x_q^N = x_q^N \left[1 + \frac{p_N}{x_q^N} \frac{\partial x_q^N}{\partial p_N} \right] \tag{3.61}$$

In order to give meaning to the right hand side of 3.61, suppose a single product monopolist exists. Since we are considering quantity sold to be a function of price, let the monopolist's demand curve be written as

$$q = f(p) \tag{3.62}$$

Total revenue is then equal to

$$\text{T.R.} = p \cdot q \tag{3.63}$$

Marginal revenue with respect to price, or how much revenue changes in response to a price change, is equal to

$$\frac{d(T.R.)}{dp} = \frac{d(p \cdot q)}{dp} = p\,\frac{dq}{dp} + q = q\left[1 + \frac{p}{q}\frac{dq}{dp}\right] \qquad (3.64)$$

It can be seen that the last expression in 3.64 is equal to the last expression in 3.61, and that both equal marginal revenue. In light of this, the condition for profit maximization can be interpreted as requiring equality between marginal revenue, but in this case the marginal revenue of a price change, and a comprehensive marginal cost figure. Profit maximization requires that what a price change adds to revenue just equals what that price change adds to costs, where costs here include variable costs, price offer variation costs, and the costs of different utilization rates of the fixed factors.

3.51 expresses the condition for the profit maximizing adjustment of a nonprice offer variation. Consider optimal adjustment of x_a^m with respect to the N^{th} good. Casting the expression into a profit margin framework, one obtains

$$\left[p_N - \sum_{k=1}^{K} \frac{\partial V}{\partial \hat{y}_k}\frac{\partial \hat{y}_k}{\partial x_a^m}\right]\frac{\partial x_q^N}{\partial x_a^m} \leq \sum_{k=1}^{K} \frac{\partial V}{\partial \hat{y}_k}\frac{\partial \hat{y}_k}{\partial x_a^m} - \sum_{n=1}^{N-1}\left[p_n - \sum_{k=1}^{K}\frac{\partial V}{\partial \hat{y}_k}\frac{\partial \hat{y}_k}{\partial x_q^n}\right]\frac{\partial x_q^n}{\partial x_a^m}$$

$$+ \sum_{s=1}^{S} \lambda_s \sum_{k=1}^{K} \frac{\partial h_s}{\partial \hat{y}_{sk}}\frac{\partial \hat{y}_{sk}}{\partial x_a^m} \qquad (3.65)$$

Upon putting this on a per unit basis with respect to the N^{th} good, the following expression results:

$$\left\{p_N - \sum_{k=1}^{K} \frac{\partial V}{\partial \hat{y}_k} \frac{\partial \hat{y}_k}{\partial x_q^N}\right\} \leq \left[\frac{\sum_{k=1}^{K} \frac{\partial V}{\partial \hat{y}_k} \frac{\partial \hat{y}_k}{\partial x_a^m} - \sum_{n=1}^{N-1} \left\{p_n - \sum_{k=1}^{K} \frac{\partial V}{\partial \hat{y}_k} \frac{\partial \hat{y}_k}{\partial x_q^n}\right\} \frac{\partial x_q^n}{\partial x_a^m}}{\frac{\partial x_q^N}{\partial x_a^m}}\right]$$

$$+ \left[\frac{\sum_{s=1}^{S} \lambda_s \sum_{k=1}^{K} \frac{\partial h_s}{\partial \hat{y}_{sk}} \frac{\partial \hat{y}_{sk}}{\partial x_a^m}}{\frac{\partial x_q^N}{\partial x_a^m}}\right] \tag{3.66}$$

3.66 states the relationship that should exist between the amount that a unit of the N^{th} good contributes to profit and the effect of this unit on costs after optimal adjustment of the m^{th} nonprice offer variation.

The left side of 3.66 is the profit margin on the N^{th} good. The first term on the right is the multiproduct nonprice offer variation cost from the Scitovsky-Holdren model. The second term on the right is also a cost of production--the cost of using fixed factors. $\sum_{k=1}^{K} \frac{\partial h_s}{\partial \hat{y}_{sk}} \frac{\partial \hat{y}_{sk}}{\partial x_a^m}$ represents the total change in the utilization of fixed factor s because more variable inputs were hired in response to changing x_a^m. λ_s is the objectively determined valuation of an additional unit of fixed factor s. The dollar value of the change in the level of employment of fixed factor s resulting from changing x_a^m can therefore be represented by $\lambda_s \sum_{k=1}^{K} \frac{\partial h_s}{\partial \hat{y}_{sk}} \frac{\partial \hat{y}_{sk}}{\partial x_a^m}$, and the sum over all s is the total valuation of any change in fixed factor utilization caused by changing x_a^m. Division by $\frac{\partial x_q^N}{\partial x_a^m}$ once again puts this on a per unit basis.

In total, 3.66 says that in order to utilize x_a^m in the profit maximizing manner, the firm should employ it to where that aspect of the firm's offer causes the profit margin on the N^{th} good to just exactly cover costs incurred by producing and selling the last item of the N^{th} good. In other words, what x_a^m adds to net revenue (profit) for any good is just offset by the additional cost of using x_a^m. The strict equality must hold in 3.66 in order to use x_a^m.

If the inequality were to hold, what x_a^m adds to cost exceeds its profit contribution with respect to the N^{th} good. This fact, in conjunction with 3.52, requires that x_a^m not be used and 3.53 permits a zero value for any x_a^m.

The remaining conditions, 3.54, 3.55, and 3.56 all relate to the utilization of the fixed factors. Rearrangement of 3.54 results in

$$h_s(\hat{y}_{s1}, \hat{y}_{s2}, \ldots, \hat{y}_{sK}) \leq \bar{y}_s \quad (3.54a)$$

$$s = 1, 2, \ldots, S$$

which is simply the earlier condition that utilization of any fixed factor cannot exceed its availability. If the strict inequality holds in 3.54a, then $\overset{\circ}{\lambda}_s = 0$ because of 3.55. Since λ_s is the value of an additional unit of fixed factor s, $\overset{\circ}{\lambda}_s = 0$ in such a case because there is already a surplus amount of factor s and additional units are unneeded and add nothing to the value of the firm. Finally, the imputed value of an additional unit of any fixed factor is restricted to be non-negative.

Expressions 3.59 and 3.66 are analogous to the equations of the Holdren model, 2.13 and 2.14. The additional term relating to the fixed factors is missing from the earlier equations because it was assumed in that model that no fixed factors were being used to capacity. This case is handled in 3.59 and 3.66 because if no fixed factors are being completely used, all λ_s would equal zero, and then the equations would almost be identical.

They aren't identical because of the problem that was alluded to earlier. Conditions 3.48 to 3.56 give rise to conditions for the profit maximizing levels of prices and non-price offer variations. Corresponding to these will be levels of outputs from all types of processes: final sold products, intermediate products, and nonsold outputs. Associated with these levels of outputs are costs. In Equations 2.13 and 2.14 the C-function of total cost was assumed to represent the least cost method of production of each level of output. We have not shown that the value which $V(\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_K)$ takes on in 3.59 or 3.64 represents the least cost method of production for the output levels of all processes which result from selecting the appropriate values of p_n and x_a^m. To get this minimum cost, we can go back and make use of the side relations, 3.25, which are the statements of the technical relationships of the processes. The output of each process has been fixed at some level, x^{i°, which corresponds to optimal adjustment of all prices and non-price offer variations. The firm now tries to minimize cost subject to the requirement that the outputs of processes are at these levels. Formally the problem is to minimize

$$V(\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_K) \tag{3.67}$$

subject to

$$\begin{aligned} g_1(\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_K; x_y^{21}, x_y^{31}, \ldots, x_y^{R1}) &= x^{1^\circ} \\ g_2(\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_K; x_y^{12}, x_y^{32}, \ldots, x_y^{R2}) &= x^{2^\circ} \qquad (3.68) \\ &\vdots \\ g_{R+N+M}(\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_K; x_y^{1,R+N+M}, \ldots, x_y^{R,R+N+M}) &= x^{R+N+M^\circ} \end{aligned}$$

This problem can be handled in the classical constrained optimization framework. The Lagrangian function would be

$$L(\hat{y}_k, u_s) = V(\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_K) + \sum_{j=1}^{R+N+M} u_j [x^{j^\circ} - g_j(\hat{y}_k; x_y^{rj})] \tag{3.69}$$

First order conditions for the least cost combination of inputs requires that

$$\frac{\partial L}{\partial \hat{y}_k} = \frac{\partial V}{\partial \hat{y}_k} - \sum_{j=1}^{R+N+M} u_j \frac{\partial g_j}{\partial \hat{y}_k} = 0 \tag{3.70}$$

$$k = 1, 2, \ldots, K$$

Rearrangement of 3.70 results in

$$\frac{\partial V}{\partial \hat{y}_k} = \sum_{j=1}^{R+N+M} u_j \frac{\partial g_j}{\partial \hat{y}_k} \tag{3.71}$$

$$k = 1, 2, \ldots, K$$

$\frac{\partial V}{\partial \hat{y}_k}$ represents the change in cost that occurs because an additional unit of $\hat{y}_k$ is purchased. $\frac{\partial g_j}{\partial \hat{y}_k}$ is the marginal product of the j^{th} process with respect to the k^{th} input. u_j is the Lagrangian multiplier and can be interpreted as the dollar value of an additional unit of output of the j^{th} process.

The sum $\sum_{j=1}^{R+N+M} u_j \frac{\partial g_j}{\partial \hat{y}_k}$ is thus the total value of additional output from all processes which results from employing more $\hat{y}_k$. For the least cost combination of inputs, 3.71 says that an input should be hired to where what it adds to cost just equals what it adds to the value of output.

CHAPTER IV. APPLICATIONS OF THE MULTIPRODUCT MODEL AND COMMENTS ON INTERPRETATION

The first part of this chapter discusses several ways of using this model in gaining insight into a firm's short run decisions. The distinguishing characteristic of this short run is its immediacy. In this short run the concern of the firm is what it must do in order to sell something. It must make decisions affecting its offer. This discussion is by no means exhaustive. The model is very general and by appropriately defining the variables one can make it applicable to a wide array of situations. The chapter ends by reiterating exactly what the objective of the model is.

Applications of the Model

Any change that a firm makes in its offer may affect both revenues and costs. The effects on revenues will include such things as the changes in the levels of sales of products, widths of profit margins, and the levels of nonprice offer variations. The effects on costs relate to such things as utilization rates of fixed factors, output rates of processes, and the choice of processes actually used. These costs arise from changes in the employment levels of the variable factors and their allocation within the firm.

Consider first the effects on sales levels and profit margins of some change in the firm's offer. Assume the firm sells N products and that each product has its own downward sloping sales function:

$$
\begin{aligned}
x_q^1 &= x_q^1(p_1, p_2, \ldots, p_N; A) \\
&\vdots \\
x_q^N &= x_q^N(p_1, p_2, \ldots, p_N; A)
\end{aligned}
\tag{4.1}
$$

where

$$\frac{\partial x_q^i}{\partial p_i} < 0 \qquad i = 1, 2, \ldots, N \tag{4.2}$$

and A represents the vector of nonprice offer variations.

To simplify the discussion of costs, consider a surrogate total cost function

$$\text{T.C.} = C(x_q^1, x_q^2, \ldots, x_q^N; A) \tag{4.3}$$

which expresses costs simply as a function of the levels of final outputs.

Profit is the difference between total revenue and total cost and can be represented as

$$\pi = \sum_{n=1}^{N} p_n\, x_q^n - C(x_q^1, x_q^2, \ldots, x_q^N; A) \tag{4.4}$$

Concentrating on the profit maximizing adjustment of prices, first order conditions require that

$$\frac{\partial \pi}{\partial p_n} = p_n \frac{\partial x_q^n}{\partial p_n} + x_q^n + \sum_{\substack{i=1 \\ i \neq n}}^{N} p_i \frac{\partial x_q^i}{\partial p_i} - \sum_{i=1}^{N} \frac{\partial C}{\partial x_q^i} \frac{\partial x_q^i}{\partial p_n} = 0 \qquad (4.5)$$

for all prices. Note that this is forced to equal zero because of the previous stipulation that prices must be positive.

Using good N as representative, putting 4.5 into the profit margin format results in

$$p_N - \frac{\partial C}{\partial x_q^N} = -x_q^N \Big/ \frac{\partial x_q^N}{\partial p_N} - \sum_{n=1}^{N-1} \left(p_n - \frac{\partial C}{\partial x_q^n} \right) \frac{\partial x_q^n}{\partial p_N} \Big/ \frac{\partial x_q^N}{\partial p_N} \qquad (4.6)$$

By looking at 4.6 one can see that the signs of $\frac{\partial x_q^n}{\partial p_N}$ are going to have a substantial influence on the profit margin of the N^{th} good. Other goods sold by the firm can be ignored only if good N is independent of all other sold outputs. If this were true, then

$$\frac{\partial x_q^n}{\partial p_N} = 0 \qquad (4.7)$$

$$n = 1, 2, \ldots, N-1$$

and the firm could adjust p_N oblivious of the effect good N has on the sales levels of the other goods.

However, if the $\frac{\partial x_q^n}{\partial p_N}$ terms are nonzero, these other products must be considered. If the relationship of good N with the other goods is one that is dominantly substitution, then the $\frac{\partial x_q^n}{\partial p_N}$ which are positive will on balance influence the second term on the right side of 4.6. This fact makes the right side larger (It is assumed that the weighted profit margin term is positive for each good.), implying that the firm should adjust p_N towards a relatively high profit margin.

From an intuitive standpoint as to what the firm should do, this makes sense. If a firm sells products which are substitutes, the fact that a customer buys one of those products precludes his buying another. Thus if the firm is going to make a sale, it should try to make that one sale as profitable as possible.

Alternatively, if good N is dominantly complementary with all other goods that this firm sells, then those $\frac{\partial x_q^n}{\partial p_N}$ which are negative will be the governing influence in the second term. This makes the second term on the right side of 4.6 positive, and when this is subtracted from the positive first term, it reduces the profit margin on good N.

Once again, from an intuitive standpoint this makes sense. If all of the sold outputs are dominantly complements, they are related in sales and/or use. If the firm can sell good N to a particular customer, perhaps the customer will also buy items to use with good N. It might benefit the firm to aim for a smaller profit margin on good N because this small margin will be more than compensated for by selling additional items.

The above interpretation has some interesting implications if the goods represent sales at different points in time instead of different products. Suppose the firm submits bids for various contracts. If it is thought that several contracts might be forthcoming over time, the firm could consider such contracts complementary goods, because if it performs well on one contract it might have an advantage getting later orders. In light of the results of the model, the firm might have an incentive to aim for a relatively small profit margin on the first contract in anticipation of additional profits later.

From a buyer's standpoint, this also gives rise to a strategy for making purchases. Even if the buyer knows that he will only need to purchase some item once, it may be wise to announce that this is only the first of potentially many purchases. If suppliers believe this, their smaller profit margins could leave the buyer much better off.

The explicitness of the model can also be useful in showing more rigorously some of the results from conventional analysis of the firm. Suppose the firm is a monopolist selling only one product, but that it sells this product in two different markets. The problem facing the firm is how to divide its output between the two markets in such a manner as to maximize profit.

Let the output that is sold in the two markets be denoted as x_q^1 and x_q^2. The firm then faces a sales function in each market:

$$\begin{aligned} x_q^1 &= x_q^1(p_1, p_2; A) \\ x_q^2 &= x_q^2(p_1, p_2; A) \end{aligned} \tag{4.8}$$

Cost will be given as the surrogate cost function:

$$T.C. = C(x_q^1, x_q^2, A) \tag{4.9}$$

Profit is the difference between total revenue and total cost and is given by

$$\pi = p_1 x_q^1 + p_2 x_q^2 - C(x_q^1, x_q^2; A) \tag{4.10}$$

First order conditions for profit maximization require that

$$\frac{\partial \pi}{\partial p_1} = p_1 \frac{\partial x_q^1}{\partial p_1} + x_q^1 + p_2 \frac{\partial x_q^2}{\partial p_1} - \frac{\partial C}{\partial x_q^1} \frac{\partial x_q^1}{\partial p_1} - \frac{\partial C}{\partial x_q^2} \frac{\partial x_q^2}{\partial p_1} = 0$$

$$\frac{\partial \pi}{\partial p_2} = p_2 \frac{\partial x_q^2}{\partial p_2} + x_q^2 + p_1 \frac{\partial x_q^1}{\partial p_2} - \frac{\partial C}{\partial x_q^1} \frac{\partial x_q^1}{\partial p_2} - \frac{\partial C}{\partial x_q^2} \frac{\partial x_q^2}{\partial p_2} = 0 \tag{4.11}$$

Putting these into the marginal revenue-marginal cost framework of 3.60, one obtains:

$$p_1 \frac{\partial x_q^1}{\partial p_1} + x_q^1 = \frac{\partial C}{\partial x_q^1} \frac{\partial x_q^1}{\partial p_1} + \left(p_2 - \frac{\partial C}{\partial x_q^2}\right) \frac{\partial x_q^2}{\partial p_1}$$

$$p_2 \frac{\partial x_q^2}{\partial p_2} + x_q^2 = \frac{\partial C}{\partial x_q^2} \frac{\partial x_q^2}{\partial p_2} + \left(p_1 - \frac{\partial C}{\partial x_q^1}\right) \frac{\partial x_q^1}{\partial p_2} \qquad (4.12)$$

The left sides of these expressions are the marginal revenues with respect to price of good one and good two, respectively. $\frac{\partial C}{\partial x_q^1} \frac{\partial x_q^1}{\partial p_1}$ and $\frac{\partial C}{\partial x_q^2} \frac{\partial x_q^2}{\partial p_2}$ are the marginal cost of additional units of these goods. Since the goods are identical in production and only distinguishable because they are sold in different markets, these marginal cost of production expressions will be equal:

$$\frac{\partial C}{\partial x_q^1} \frac{\partial x_q^1}{\partial p_1} = \frac{\partial C}{\partial x_q^2} \frac{\partial x_q^2}{\partial p_2} = \text{M.C.} \qquad (4.13)$$

We can now show how crucial is an assumption made in this analysis in order to get the standard result that the firm should allocate output so that the marginal revenues in each market are equal to each other, and that these are equal to marginal cost. In order to practice this type of price discrimination, it is necessary to isolate the markets. This means that the seller must keep the customers separate so that arbitrage cannot take place. If he is capable of doing this, then

$$\frac{\partial x_q^1}{\partial p_2} = \frac{\partial x_q^2}{\partial p_1} = 0 \qquad (4.14)$$

and the last term on the right sides of the expressions in 4.12 disappear. A simple collection of terms then gets the familiar result:

$$M.R._1 = M.R._2 = M.C. \qquad (4.15)$$

On the production side, the interdependencies that are of interest can be explicitly dealt with. The partial derivatives $\frac{\partial h_s}{\partial x_q^n}$ and $\frac{\partial h_s}{\partial x_a^m}$ are to be viewed in a comprehensive nature. The rate of utilization of fixed factor s will of course change because x_q^n changes, but the change in x_q^n could also affect the output levels of all processes. These changes will also affect the utilization of factor s. The same chain of events must be kept in mind when some x_a^m is changed. This change could again conceivably affect the output rates of all other processes, and these changes will again affect the utilization of fixed factor s.

The model also recognizes the problem of process choice and establishes criteria. For simplicity suppose the firm consists of only one process, but that two different technologies, a and b, exist for accomplishing this process. Also assume that the firm buys its inputs in perfectly competitive markets.

If the firm uses technology a, a condition for least cost combination of inputs corresponding to 3.70 would be

$$v_k = u_a \frac{\partial g_a}{\partial \hat{y}_k} \tag{4.16}$$

$$k = 1, 2, \ldots, K$$

where v_k is the price of input k.

If the firm uses technology b, the analogous relationship would be

$$v_k = u_b \frac{\partial g_b}{\partial \hat{y}_k} \tag{4.17}$$

$$k = 1, 2, \ldots, K$$

Putting 4.16 and 4.17 together, the following relation must hold if the firm is using two (or more) different procedures to carry out a given process:

$$u_a \frac{\partial g_a}{\partial \hat{y}_k} = u_b \frac{\partial g_b}{\partial \hat{y}_k} \tag{4.18}$$

This states that what an input adds to the value of output in one technology should equal what that input adds to the value of output if it is used in another technology. If the equality did not hold, units of the input should be transferred from the technology with the smaller value to the technology with the larger value.

This valuation of processes has some usefulness when one starts talking about acquisition criteria when a firm is considering growth or expansion. Any given firm is a collection of processes and any firm that is a take-over target is also a collection of processes. By being able to assign value to the outputs of the various processes, the firm is capable of evaluating different combinations of processes, where the processes come from either firm or are new processes which result from the combining of the two firms' assets.

This simplified firm with a small number of products and processes can also be used to show more lucidly the interpretation of the Lagrangian multipliers, λ_s and u_i. Suppose that the firm sells two products, x_q^1 and x_q^2, and that the firm is composed of three processes. Profit maximizing adjustment of p_1 results in

$$p_1 - \sum_{k=1}^{K} \frac{\partial V}{\partial \hat{y}_k} \frac{\partial \hat{y}_k}{\partial x_q^1} = - x_q^1 \Big/ \frac{\partial x_q^1}{\partial p_1} - \left(p_2 - \sum_{k=1}^{K} \frac{\partial V}{\partial \hat{y}_k} \frac{\partial \hat{y}_k}{\partial x_q^2} \right) \frac{\partial x_q^2}{\partial p_1} \Big/ \frac{\partial x_q^1}{\partial p_1}$$

$$+ \sum_{s=1}^{S} \lambda_s \sum_{k=1}^{K} \frac{\partial h_s}{\partial \hat{y}_{sk}} \frac{\partial \hat{y}_{sk}}{\partial x_q^1} \qquad (4.19)$$

If we assume the two products are independent, $\frac{\partial x_q^2}{\partial p_1}$ is zero, and 4.19 reduces to

$$p_1 - \sum_{k=1}^{K} \frac{\partial V}{\partial \hat{y}_k} \frac{\partial \hat{y}_k}{\partial x_q^1} = - x_q^1 \Big/ \frac{\partial x_q^1}{\partial p_1} + \sum_{s=1}^{S} \lambda_s \sum_{k=1}^{K} \frac{\partial h_s}{\partial \hat{y}_{sk}} \frac{\partial \hat{y}_{sk}}{\partial x_q^1} \qquad (4.20)$$

The left side of 4.20 is the profit margin on good one. The first term on the right is the price offer variation cost. The second term is the imputed value of the fixed factors. If no fixed factors are being used to capacity, then all λ_s are zero and this term vanishes. However, as soon as some fixed factor is used to capacity, additional units of this factor would have value. This means that the λ associated with this factor, which we stated represents the value of additional units of the factor, becomes positive. When this happens, the profit margin must cover not only price offer variation cost but also must pay for the use of the fixed factor. This is true because if a fixed factor is being used to capacity, the use of some of it for good one may limit the amount that is available for use in good two. Thus an opportunity cost of using the factor arises which must be recovered. The positive value of the λ reflects this cost.

The small number of processes also permits interpretation of the u_i's. Recall that least cost production requires that

$$\frac{\partial V}{\partial \hat{y}_k} = \sum_{i=1}^{R+N+M} u_i \frac{\partial g_i}{\partial \hat{y}_k} \qquad (4.21)$$

Suppose that input k is used only in good one, and that good one requires only process one. 4.21 becomes

$$\frac{\partial V}{\partial \hat{y}_k} = u_1 \frac{\partial g_1}{\partial \hat{y}_k} \qquad (4.22)$$

Since only one process is used, this situation is analogous to models of the firm that postulate the firm possesses a production function. 4.22 exactly parallels the result from these models that the least cost production of a given level of output requires that an input be used to where the amount it adds to cost is equal to the value of its marginal product. Isolating u_1, one obtains

$$\frac{\partial V}{\partial \hat{y}_k} \bigg/ \frac{\partial g_1}{\partial \hat{y}_k} = u_1 \qquad (4.23)$$

This ratio is the dollar value per unit of output of process one. This corresponds entirely with our earlier interpretation of the u_i's.

Comments on Interpretation

The model was constructed so that the goal of the firm was to maximize profit at some arbitrary point in time. As such it represents part of the overall problem facing the firm: the maximizing of profit over time, $\int_0^\infty \pi \, dt$. In order to maximize $\int_0^\infty \pi \, dt$, it is necessary to maximize profit at every time t.

The model developed in Chapter III does exactly that for an arbitrary t. Since we can solve the model for any t, we can do it for every t. Thus the results of the problem in Chapter III represent a necessary point on the path of profit that maximizes $\int_0^\infty \pi \, dt$.

This maximization of profit over time represents a control problem. In the generalized control problem, attention is centered on some real or hypothetical system. In the case at hand, that system is the firm. The problem is that of optimizing the behavior of the system through time by choosing the time paths of certain variables called control variables. This represents a powerful approach. It forces us to realize that we can't have a simple one period maximization. One period can't be looked at in isolation because what has transpired in previous periods affects the current period and the current period affects future periods. Related to this is the fact that it forces upon us more clearly the distinction between those entities which we treat as variables and those entities which we treat as parameters at any given decision point.

At time t, the driving force is the maximization of profit over sold output. The variables are those things which affect the firm's offer: prices and nonprice offer variations. In response to these variables, we will get output levels of sold products and intermediate products. This represents the minimum set of decisions the firm must make in order to maximize profit. The levels of prices and some nonprice offer variations represent the unavoidable, initial decisions made at time t.

Parameters at time t are those other entities over which the firm has either no control or whose values are not arrived at via the profit maximization at time t route. As an example of the former, the firm has no control over the stock of fixed factors it inherits from previous periods. For an example of the latter, the levels of the x_I^w's are not determined by the profit maximizing conditions in Chapter III. Included in this would be investment and research and development activities whose costs and benefits may not be measurable at time t, and also some nonprice offer variation type activities whose levels are not alterable at time t (i.e., a long term television contract). These x_I^w's are variables in the overall problem but at time t they are parameters, fixed at levels which the firm takes as given. These levels are important to profit maximization at time t because they affect the amounts of resources available to the firm over which to maximize.

Getting back to relating our current problem to the control problem, we can describe the manner in which the firm changes through time by specifying the time behavior of a finite number of variables $y_1(t), \ldots, y_n(t)$. These are called state variables. For the firm they might include capital stock, assets, work force, technical expertise, reputation, etc.

There also exists a set of control variables $v_1, v_2, \ldots, v_m$. These are such that if the time paths of the control variables are specified, then the time variation of the state variables is determined. Control variables for the firm in the short run are prices and some nonprice offer variations. A control is a vector valued function of time whose components are the control variables.

To complete the problem, one needs a definition of the effectiveness of control. Such a measure is provided by

$$J[\vec{y}, \vec{v}] = \int_{t_o}^{t_1} F(t, \vec{y}(t), \vec{v}(t))\, dt \qquad (4.24)$$

where $\vec{}$ above a letter denotes a vector.

The control problem is then that of choosing a control $\vec{v}(t)$ from the set of allowable values for each t such that when $\vec{y}(t)$ is determined and an initial value is given, the functional 4.24 is maximized or minimized. A control which satisfies these conditions is called an optimal control.

For the firm under consideration, 4.24 would be the profit function, where profit at any time depends on the values of the state variables and the control variables. It is the choosing of the values of the control variables that is addressed in Chapter III. At a point in time the firm chooses the levels of prices and nonprice offer variations to maximize profit. By doing this for every time t, the expression in 4.24 is maximized.[1]

We are still left with a few gaps in the model. In particular, the choice of the set of the x_q^n's and the set of the x_a^m's has never been mentioned. Also, some of the x_I^w's represent investment projects or other discrete activities, the selection of which requires special handling. Prior to the maximization of profit, the firm has to solve an integer programming problem, the solution of which is a discrete number of activities in which the firm should engage. The problem in Chapter III solves for levels of utilization of processes, but not for the exact array of processes. Integer programming is needed because these activities come in indivisible units. A partial product, a fraction of a nonprice offer variation, or part of an investment project would be meaningless entities.[2]

A final qualification of the model results from the fact that we simply do not know enough about how business firms behave in order to be sure that we are asking the right questions. The profit maximization assumption may be more

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[1]This discussion of optimal control comes from (19, pp. 238-242). It is couched in continuous terms as was the pioneering work by Pontryagin et al. (31). In solving such problems the necessary conditions are referred to as the maximum principle. In the case of a business firm, accounting procedures and other practicalities require a discrete analogue and one has been derived (16). For a brief but good discussion of optimal control applied to economics, see (11).

[2]For a discussion of situations where it is necessary to use integer programming, see (6).

heroic than it seems. It is questionable whether the firm is knowledgeable enough to maximize profit because it operates in an atmosphere of so many unknowns. It may not know all of the decision variables available to it, and even if it does it may not know the effect of manipulating these decision variables.

These issues represent not so much limitations of the model as questions still to be answered. As the model was developed, it represents a general approach to optimization by a multiproduct firm. Remaining problems arise not so much in the fact that the questions aren't answered as in the fact they aren't asked.

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